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# The double square root, Jacobi polynomials and Ramanujan's Master Theorem

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**Abstract**

Let

$$N_{0,4}(a;m) := \int_0^\infty \frac{dx}{(x^4+2ax^2+1)^{m+1}}, \quad a > -1, \ m \in \mathbb{N}$$

and define

$$P_m(a) := \frac{1}{\pi} 2^{m+3/2}(a+1)^{m+1/2} N_{0,4}(a;m).$$

We prove that $P_m(a)$ is a polynomial in $a$ given by

$$P_m(a) = 2^{-2m} \sum_{k=0}^{m} 2^k \binom{2m-2k}{m-k} \binom{m+k}{m} (a+1)^k.$$

The proof is based on the Taylor expansion of the double square root and Ramanujan's Master Theorem. 

*MSC:* primary 33

*Keywords:* Rational functions; Integrals; Double square root; Snake oil method

## 1. Introduction

The standard tables of integrals [5,8] contain very few explicit evaluations of definite integrals of rational functions of degree bigger than 2. The quartic integral

$$N_{0,4}(a;m) := \int_0^\infty \frac{dx}{(x^4+2ax^2+1)^{m+1}} \tag{1.1}$$

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is implicit in [5, 3.252.11] and is expressed there in terms of hypergeometric functions. We have presented in [3] a proof of the identity

$$N_{0,4}(a;m) = \frac{\pi}{2^{m+3/2}(a+1)^{m+1/2}} \times P_m(a), \tag{1.2}$$

where

$$P_m(a) = 2^{-2m} \sum_{k=0}^{m} 2^k \binom{2m-2k}{m-k} \binom{m+k}{m} (a+1)^k. \tag{1.3}$$

In this paper we give an evaluation of (1.2) that is free of hypergeometric functions. We develop a formula for the Taylor expansion of the double square root and then use Ramanujan's Master Theorem. The hypergeometric proof of (1.2) is clearly simpler but we expect to use the ideas developed here in the discussion of higher order rational integrands.

## 2. A useful integral

The formula

$$\int_0^\infty \left( \frac{u^2}{bu^4 + 2au^2 + 1} \right)^r \mathrm{d}u = \frac{B(r - \frac{1}{2}, \frac{1}{2})}{2^{r+1/2}\sqrt{b} \times (a + \sqrt{b})^{r-1/2}} \tag{2.1}$$

for $b > 0$, $a + \sqrt{b} > 0$ and $r > \frac{1}{2}$ follows directly from

$$\int_0^\infty \left( \frac{x^2}{x^4 + 2ax^2 + 1} \right)^r \mathrm{d}x = \frac{B(r - \frac{1}{2}, \frac{1}{2})}{2^{r+1/2}(1 + a)^{r-1/2}} \tag{2.2}$$

and the change of variable $u \to b^{-1/4}x$. A proof of (2.2) is given in [2].

**Proposition 2.1.** *Let* $a, b > 0$. *Then*,

$$\int_0^\infty \frac{\mathrm{d}x}{bx^4 + 2ax^2 + 1} = \frac{\pi}{2\sqrt{2}} \frac{1}{\sqrt{a + \sqrt{b}}}.$$

**Proof.** Let $x = 1/u$ in (2.1) with $r = 1$ and replace $b$ by $1/b$ and $a$ by $a/b$. □

## 3. The Taylor expansion of the double square root

We now evaluate the coefficients of the Taylor expansion of $h(c) := \sqrt{a + \sqrt{1+c}}$. The particular case $a = 1$ is a standard example often used to illustrate Lagrange's inversion formula. Berndt [1, pp. 71,72 and 304–307], gives a complete history of this problem.

**Lemma 3.1.** *For* $|c| < 1$, *define*

$$g(c) = \int_0^\infty \frac{\mathrm{d}x}{x^4 + 2ax^2 + 1 + c}$$

*and* $h(c)=\sqrt{a+\sqrt{1+c}}$. *Then* $g(c)=\pi\sqrt{2}h'(c)$. *In particular,*

$$h'(0)=\frac{1}{\pi\sqrt{2}}N_{0,4}(a;0).$$

**Proof.** Write $g(c)$ as

$$g(c)=\frac{1}{1+c}\int_0^\infty \frac{\mathrm{d}x}{x^4/(1+c)+(2a/(1+c))x^2+1}$$

and now use Proposition 2.1 to evaluate $g(c)$. □

**Theorem 3.2.** *The Taylor expansion of* $h(c)=\sqrt{a+\sqrt{1+c}}$ *is given by*

$$\sqrt{a+\sqrt{1+c}}=\sqrt{a+1}+\frac{1}{\pi\sqrt{2}}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}N_{0,4}(a;k-1)c^k. \tag{3.1}$$

**Proof.** Evaluate $h^{(k)}(0)$ using Lemma 3.1 □

**Corollary 3.3.** *We have*

$$\sqrt{1+\sqrt{1+c}}=\sqrt{2}\left[1+\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\frac{1}{2^{4k-1}}\binom{4k-3}{2k-2}c^k\right].$$

**Proof.** Use the value $2^{4k-2}N_{0,4}(1;k-1)=\pi\binom{4k-3}{2k-2}$. □

This appears in [4, p. 192, exercise 21], and is a special case of Gradshteyn and Ryzhik [5, 1.114.1].

## 4. Ramanujan's Master Theorem and a new class of integrals

We establish a connection between $N_{0,4}(a;m)$ and a new family of integrals. This is used in Section 5 to prove (1.3).

**Theorem 4.1.** *Define*

$$J_m(a):=\int_0^\infty \frac{x^{m-1}\,\mathrm{d}x}{(a+\sqrt{1+x})^{2m+1/2}}. \tag{4.1}$$

*Then,*

$$J_m(a)=\frac{1}{\pi}2^{6m+3/2}\left[m\binom{4m}{2m}\binom{2m}{m}\right]^{-1}\times N_{0,4}(a;m). \tag{4.2}$$

The proof of Theorem 4.1 is based on Ramanujan's Master Theorem stated below.

**Theorem 4.2.** *Suppose $F$ has a Taylor expansion around $c = 0$ of the form*

$$F(c) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\varphi(k)c^k.$$

*Then, the moments of $F$, defined by*

$$M_m = \int_0^\infty c^{m-1}F(c)\,\mathrm{d}c, \tag{4.3}$$

*can be computed via*

$$M_m = \Gamma(m)\varphi(-m). \tag{4.4}$$

Berndt [1] provides a proof and exact hypotheses for the validity of the Master Theorem; see also [6] for more information. Observe that expression (4.4) requires the ability to compute the function $\varphi$ outside its original range, namely at negative indices.

**Proof of Theorem 4.1.** We apply the Master Theorem to (3.1). Differentiate the integral $N_{0,4}(a; k-1)$ $j$ times and replace $x$ by $1/x$ to produce

$$\left(\frac{\mathrm{d}}{\mathrm{d}a}\right)^j N_{0,4}(a; k-1) = \frac{(-1)^j 2^j (k+j-1)!}{(k-1)!} \times \int_0^\infty \frac{x^{4k+2j-2}\,\mathrm{d}x}{(x^4 + 2ax^2 + 1)^{k+j}}.$$

From (3.1) we obtain

$$\left(\frac{\mathrm{d}}{\mathrm{d}a}\right)^j \sqrt{a + \sqrt{1+c}} = \left(\frac{\mathrm{d}}{\mathrm{d}a}\right)^j \sqrt{a+1} + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!}\varphi(k)c^k$$

with

$$\varphi(k) = (-1)^{j+1}\frac{1}{\pi\sqrt{2}}(k+j-1)!2^j \times \int_0^\infty \frac{x^{4k+2j-2}\,\mathrm{d}x}{(x^4+2ax^2+1)^{k+j}}.$$

Now replace $k$ by $-m$ to produce

$$\varphi(-m) = (-1)^{j+1}\frac{1}{\pi\sqrt{2}}(-m+j-1)!2^j \times \int_0^\infty \frac{x^{-4m+2j-2}\,\mathrm{d}x}{(x^4+2ax^2+1)^{-m+j}}.$$

The choice $j = 2m+1$ yields

$$\varphi(-m) = \frac{m!2^{2m+1}}{\pi\sqrt{2}}N_{0,4}(a; m). \tag{4.5}$$

The moments of the function $H(c) := (\mathrm{d}/\mathrm{d}a)^j\sqrt{a+\sqrt{1+c}}$ are computed directly as

$$M_k = \frac{(-1)^{j+1}(2j-3)!}{2^{2(j-1)}(j-2)!} \times \int_0^\infty \frac{c^{k-1}\,\mathrm{d}c}{(a+\sqrt{1+c})^{j-1/2}}$$

and the choices $j = 2m+1$ and $k = m$ produce

$$M_m = \frac{(4m-1)!}{2^{4m}(2m-1)!} \times J_m(a).$$

Ramanujan's Master Theorem now yields (4.2).

## 5. A simplified expression for $P_m(a)$

We use (4.2) to prove that

$$P_m(a) := \frac{1}{\pi} 2^{m+3/2}(a+1)^{m+1/2} N_{0,4}(a;m) \tag{5.1}$$

is a polynomial in $a$. The proof identifies $P_m(a)$ as a Jacobi polynomial with parameters $m+\frac{1}{2}, -(m+\frac{1}{2})$.

**Proposition 5.1.** *The integral $J_m(a)$ in (4.1) is given by*

$$J_m(a) = \frac{2^{2m+1}(2m)!}{(4m)!} \sum_{j=0}^{m} 2^{2j} \frac{(2m-2j)!}{(m-j)!} f_m^{(j+m-1)}(1) \times (1+a)^{-(2m-2j+1)/2},$$

*where*

$$f_m(u) = u(u^2-1)^{m-1}.$$

**Proof.** The substitution $u = \sqrt{1+x}$ yields

$$J_m(a) = 2\int_1^\infty f_m(u)(a+u)^{-(2m+1/2)}\,\mathrm{d}u. \tag{5.2}$$

The result now follows by repeated integration by parts. The derivatives $f_m^{(j)}(u)$ vanish identically for $j \geqslant 2m$, and they also vanish at $u=1$ for $0 \leqslant j \leqslant m-2$. □

**Proposition 5.2.** *The polynomial $P_m(a)$ is given by*

$$P_m(a) = \frac{m}{2^{3m-1}(m!)^2} \sum_{k=0}^{m} 2^{2k} \frac{(2m-2k)!}{(m-k)!} f_m^{(k+m-1)}(1) \times (1+a)^k.$$

**Proof.** Substitute the formula in Proposition 5.1 into (4.2) and use (5.1). □

We now find a closed form for the derivatives of $f_m$ at $u=1$.

**Proposition 5.3.** *Let $0 \leqslant k \leqslant m$. Then,*

$$f_m^{(k+m-1)}(1) = 2^{m-k-1} \frac{(m-1)!(m+k)!}{(m-k)!k!}. \tag{5.3}$$

**Proof.** Expanding $f_m(u)$ and differentiating we have

$$f_m^{(k+m-1)}(1) = \sum_{j\geqslant 0} (-1)^j \binom{m-1}{j} \times \frac{(2m-2j-1)!}{(m-2j-k)!}.$$

It suffices to prove

$$b_{m,k} := \sum_{j \geqslant 0} (-1)^j \binom{m-1}{j} \binom{2m-2j-1}{k+m-1} = 2^{m-k-1} \binom{m}{k} \left(1+\frac{k}{m}\right), \tag{5.4}$$

which is equivalent to (5.3). Indeed

$$\begin{aligned}
\sum_k b_{m,k}x^{k+m-1} &= \sum_k \left( \sum_{j\geqslant 0} (-1)^j \binom{m-1}{j} \binom{2m-2j-1}{k+m-1} \right) x^{k+m-1} \\
&= \sum_{j\geqslant 0} (-1)^j \binom{m-1}{j} \sum_k \binom{2m-2j-1}{k+m-1} x^{k+m-1} \\
&= \sum_{j\geqslant 0} (-1)^j \binom{m-1}{j} \sum_k \binom{2m-2j-1}{k} x^{k} \\
&= \sum_{j\geqslant 0} (-1)^j \binom{m-1}{j} (x+1)^{2m-2j-1} \\
&= (x+1)^{2m-1} \sum_{j\geqslant 0} (-1)^j \binom{m-1}{j} (x+1)^{-2j} \\
&= (x+1)^{2m-1} \times [1-(x+1)^{-2}]^{m-1} \\
&= x^{m-1}(x+2)^m - x^{m-1}(x+2)^{m-1} \\
&= \sum_{k=0}^{m} \binom{m}{k} 2^{m-k-1} \left(1+\frac{k}{m}\right) x^{k+m-1}.
\end{aligned}$$

Thus, (5.4) holds and the proof is complete. □

**Note**. An alternative proof of (5.4) written now in the form

$$2^{-m+k+1} \binom{m}{k}^{-1} \left(1+\frac{k}{m}\right)^{-1} \sum_{j\geqslant 0} (-1)^j \binom{m-1}{j} \binom{2m-2j-1}{k+m-1} = 1 \tag{5.5}$$

is obtained using the methods developed by Wilf and Zeilberger [7]. Let $F_k(m,j)$ be the *summand* in (5.5). The WZ-algorithm produces the rational function

$$R_k(m,j) = \frac{-2j(2m+1-2j)}{(m+1+k)(m+1-2j-2k)}$$

with the property that $G_k(m,j) := R_k(m,j)F_k(m,j)$ satisfies

$$F_k(m+1,j) - F_k(m,j) = G_k(m,j+1) - G_k(m,j).$$

Summing over all $j$ we conclude that the sum of $F_k(m,j)$ is independent of $m$ and thus identically 1 as required.

**Theorem 5.4.** *The polynomial $P_m(a)$ is given by*

$$P_m(a) = 2^{-2m} \sum_{k=0}^{m} 2^k \binom{2m-2k}{m-k} \binom{m+k}{m} (a+1)^k. \tag{5.6}$$

**Proof.** This follows directly from Propositions 5.2 and 5.3. □

Expression (5.6) confirms that $P_m(a)$ is part of the Jacobi family

$$P_m^{(\alpha,\beta)}(a) := \sum_{k=0}^{m} (-1)^{m-k} \binom{m+\beta}{m-k} \binom{m+k+\alpha+\beta}{k} \left(\frac{a+1}{2}\right)^k$$

with parameters $\alpha = m + \frac{1}{2}$ and $\beta = -(m + \frac{1}{2})$.

**Corollary 5.5.** *The integral* $N_{0,4}(a;m)$ *in* (1.1) *is given by*

$$N_{0,4}(a;m) = \frac{\pi}{2^{3m+3/2}(1+a)^{m+1/2}} \sum_{k=0}^{m} 2^k \binom{2m-2k}{m-k} \binom{m+k}{m} (a+1)^k.$$

**Corollary 5.6.** *The integral* $J_m(a)$ in (4.1) *is given by*

$$J_m(a) = 2^{3m+1} \frac{m!(m-1)!(2m)!}{(4m)!(1+a)^{m+1/2}} \sum_{k=0}^{m} 2^k \binom{2m-2k}{m-k} \binom{m+k}{m} (a+1)^k.$$

**Corollary 5.7.** *The expansion of the double square root* $\sqrt{a+\sqrt{1+c}}$ *is given by*

$$\sqrt{a+\sqrt{1+c}} = \sqrt{a+1} \times \left(1 + \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} \sum_{j=0}^{k} 2^{j-3k-2} \binom{2k-2j}{k-j} \binom{k+j}{j} \frac{c^{k+1}}{(1+a)^{k+1-j}}\right). \tag{5.7}$$

**Proof.** Replace the result of Corollary 5.5 in (3.1). □

**Note**. A precise expression for the special case $c = a^2$ is due to Ramanujan. See [1, Corollary 2 to Entry 14]. The identity

$$(a+\sqrt{1+a^2})^n = 1 + na + \sum_{k=2}^{\infty} \frac{b_k(n)a^k}{k!}, \tag{5.8}$$

where, for $k \geqslant 2$,

$$b_k(n) = \begin{cases} n^2(n^2-2^2)(n^2-4^2)\cdots(n^2-(k-2)^2) & \text{if } k \text{ is even,} \\ n(n^2-1^2)(n^2-3^2)\cdots(n^2-(k-2)^2) & \text{if } k \text{ is odd,} \end{cases} \tag{5.9}$$

gives, in the case $n = \frac{1}{2}$, the Taylor series for $q(a) := \sqrt{a+\sqrt{1+a^2}}$. Our formula (5.7) gives a Laurent expansion for $q_1(a) := q(a)/\sqrt{a+1}$.

## References


[1] B. Berndt, Ramanujan's Notebooks, Part I, Springer, Berlin, 1985.
[2] G. Boros, V. Moll, An integral with three parameters, SIAM Rev. 40 (1998) 972–980.
[3] G. Boros, V. Moll, An integral hidden in Gradshteyn and Ryzhik, J. Comput. Appl. Math. 106 (1999) 361–368.
[4] T.J. Bromwich, An Introduction to the Theory of Infinite Series, 2nd Edition, MacMillan, New York, 1926.

[5] I.S. Gradshteyn, I.M. Ryzhik, in: A. Jeffrey (Ed.), Table of Integrals, Series and Products, 5th Edition, Academic Press, New York, 1994.
[6] G.H. Hardy, Ramanujan, Chelsea Publishing Company, New York, 1978.
[7] M. Petkovsek, H. Wilf, D. Zeilberger, $A = B$. A.K. Peters, Wellesley, MA, 1996.
[8] A.P. Prudnikov, Yu A. Brychkov, O.I. Marichev, Integrals and Series, Translated by G.G. Gould, Gordon and Breach, New York, 1990.